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Abstract

A digraph is odd-cycle-symmetric if every arc in any elementary odd directed cycle has the reverse arc. This concept arises in the context of the even factor problems, which generalize the path-matching problems. While the even factor problem is NP-hard in general digraphs, it is solvable in polynomial time for odd-cycle-symmetric digraphs. This paper provides a characterization of odd-cycle-symmetric digraphs and presents a linear time algorithm to determine whether a given digraph is odd-cycle-symmetric or not. The paper also discusses the weighted version.

1 Introduction

A directed graph (digraph) is *odd-cycle-symmetric* if every arc in any elementary odd directed cycle has the reverse arc. Odd-cycle-symmetric digraphs were introduced in the context of the *even factor* problems.

An even factor in a digraph is a collection of vertex-disjoint directed paths and even directed cycles, which is introduced by Cunningham and Geelen [2] as a generalization of a path-matching [1]. It is known that the problem of finding a maximum even factor is NP-hard in general, but solvable in polynomial time if the digraph is *weakly symmetric* [2], which is a special case of the odd-cycle-symmetric digraphs. A digraph is said to be weakly symmetric if every arc in any directed cycle has the reverse arc. We say a digraph is *symmetric* if every arc has the reverse arc. Recently, Pap [5] devised a polynomial algorithm for the even factor problem in an odd-cycle-symmetric digraph.

This paper gives a characterization of the odd-cycle-symmetric digraphs. For this purpose, we introduce the notion of a *cycle-connected* digraph. A digraph is said to be cycle-connected if it is strongly connected and its underlying graph is 2-connected. A digraph is said to be *bipartite* if its underlying graph is bipartite. Our main result (Theorem 1) asserts that a cycle-connected digraph is odd-cycle-symmetric if and only if it is symmetric or bipartite.

A digraph can be decomposed into cycle-connected components. This decomposition preserves odd-cycle-symmetry. Therefore, it follows from Theorem 1 that an odd-cycle-symmetric digraph can be decomposed into bipartite digraphs and symmetric digraphs. Since the decomposition can be done in linear time with the aid of basic graph algorithms, odd-cycle-symmetry can be recognized in linear time. Note that a weakly symmetric digraph can be decomposed into symmetric cycle-connected components. Thus the class of odd-cycle-symmetric digraphs is slightly broader than that of weakly symmetric digraphs.

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In addition, we discuss odd-cycle-symmetry of weighted digraphs. Let w be a weight function defined on the arc set of an odd-cycle-symmetric digraph G. Then (G, w) is said to be *odd-cycle-symmetric* if the sum of the weights of the arcs in any elementary odd directed cycle C is the same as that of its reverse cycle $\bar{C}$.

For an odd-cycle-symmetric weighted digraph (G, w), Király and Makai [4] presented a linear program that describes the maximum weight even factor problem, and proved the dual integrality. Takazawa [7] presented a combinatorial primal-dual algorithm to find a maximum weight even factor in an odd-cycle-symmetric weighted digraph. His algorithm also gives a constructive proof of the dual integrality.

In the same way as the unweighted case, we deal with a cycle-connected weighted digraph G. Theorem 1 implies that, if G is odd-cycle-symmetric, then it is bipartite or symmetric. If G is bipartite, then (G, w) is clearly odd-cycle-symmetric for any weight function w. We show that, for a digraph G that is not bipartite but symmetric, (G, w) is odd-cycle-symmetric if and only if there exists a function p on the vertex set such that $d(a) = p(v) - p(u)$ for each arc $a = (u, v)$ in G, where $d(a) = w(a) - w(\bar{a})$ for each arc a and its reverse arc $\bar{a}$. Odd-cycle-symmetry of a weighted digraph can be tested in linear time by checking the existence of such function p.

We conclude this section by giving some definitions and notations. In this paper, we consider digraphs with no loops and no multiple arcs. We denote by (u, v) the arc from u to v. We say $P = (v_1, a_1, \ldots, v_k, a_k, v_{k+1})$ is a *path* if $a_i = (v_i, v_{i+1})$ for $1 \leq i \leq k$. A path is said to be *even* if k is even, and *odd* if k is odd. If $v_i \neq v_j$ for $i \neq j$, then P is said to be *elementary*. We call v_1 and v_{k+1} the *end vertices* of P, and the other vertices the *interior vertices*. A *cycle* C is a path which ends at the vertex it begins with, namely, $C = (v_1, a_1, \ldots, v_k, a_k, v_1)$. If $v_i \neq v_j$ $(i \neq j,\ 1 \leq i, j \leq k)$, then C is said to be *elementary*.

For paths $P_1 = (v_1, a_1, \ldots, v_k, a_k, v_{k+1})$ and $P_2 = (v_{k+1}, a_{k+1}, \ldots, v_l, a_l, v_{l+1})$, we denote by $P_1 \cdot P_2$ the path $(v_1, a_1, \ldots, v_k, a_k, v_{k+1}, a_{k+1}, \ldots, v_{l+1})$. For a subgraph G' and a path P in a digraph G, we denote by $G' + P$ the subgraph that consists of the vertices and the arcs of G' and P.

For an arc a, we denote by $\bar{a}$ the reverse arc (if exists). For a path $P = (v_1, a_1, \ldots v_k, a_k, v_{k+1})$, the reverse path (if exists) is denoted by $\bar{P}$, that is $\bar{P} = (v_{k+1}, \bar{a_k}, v_k, \ldots, \bar{a_1}, v_1)$.

2 Odd-Cycle-Symmetry

Our main result is the following theorem.

Theorem 1. *A cycle-connected digraph G is odd-cycle-symmetric if and only if G is bipartite or symmetric.*

In order to prove Theorem 1, we use the ear decomposition of cycle-connected digraphs. Let G be a digraph, and G' a subgraph of G. We say that an elementary path P in G is an *ear* of G' if G' contains both of the end vertices of P, but no interior vertices and no arcs. An ear is said to be *proper* if its end vertices are distinct. Then the following lemma holds for cycle-connected digraphs. This lemma was shown by Grötschel [3], where cycle-connected digraphs are called strong blocks.

Lemma 2. *Let G be a cycle-connected digraph, and G' a subgraph of G with at least two vertices. If $G' \neq G$ then G' has a proper ear.*

Proof. Assume that there exist no proper ears of $G' = (V', A')$. Let $v_1, \dots, v_h \in V'$ be all vertices from which some arcs in $A \setminus A'$ leave, and S_i be the set of vertices which can be reached from v_i without using arcs of A'. Since G' has no proper ears, $V' \cap S_i = \{v_i\}$ for each i. Since G is strongly connected, v_j is reachable from any vertex in S_j. Hence $S_i \cap S_j = \emptyset$ for $i \neq j$, so $S_1, \dots, S_h$ is a partition of $(V \setminus V') \cup \{v_1, \dots, v_h\}$. Since there are no arcs between S_i and S_j for $i \neq j$, we can separate $S_i \setminus \{v_i\}$ from $V \setminus S_i$ by deleting v_i, which contradicts that the underlying graph of G is 2-connected. □

We also give a characterization of digraphs without elementary odd cycles.

Lemma 3. *A strongly connected digraph G has no elementary odd cycles if and only if G is bipartite.*

Proof. The necessity is obvious. To see the sufficiency, suppose $G = (V, A)$ has no elementary odd cycles. Then G has no odd cycles. Let $a \in A$ be an arc from u to v such that $\bar{a} \notin A$. Since G is strongly connected and has no odd cycles, there exists an odd path P from v to u. Then there exist no even paths from u to v, and hence the digraph obtained from G by adding the reverse arc $\bar{a}$ also has no odd cycles. Thus $G' = (V, A \cup \bar{A})$ has no odd cycles, which means G is bipartite. □

By these lemmas, we have the following proposition.

Proposition 4. *Let G be a cycle-connected odd-cycle-symmetric digraph that is not bipartite. There exists a sequence $G_0, G_1, \dots, G_k = G$ of subgraphs such that the following (A) and (B) hold.*

(A) G_0 consists of an elementary odd cycle C and its reverse cycle $\bar{C}$.

(B) G_{i+1} is obtained from G_i by adding P_i and $\bar{P}_i$, where P_i is a proper ear having the reverse path $\bar{P}_i$, for $i = 0, 1, \dots, k-1$.

Furthermore, the sequence $G_0, G_1, \dots, G_k$ satisfies the following (C) and (D).

(C) There exist both an elementary even path and an elementary odd path from u to v in $G_i = (V_i, A_i)$ for every vertex pair $u, v \in V_i$.

(D) If P is a proper ear of G_i, then every arc of P has the reverse arc.

Proof. By Lemma 3, G has an elementary odd cycle C. Hence there exists a subgraph G_0 satisfying (A). Since Lemma 2 and the condition (D) assure that there exists a sequence $G_0, G_1, \dots, G_k = G$ of subgraphs such that (A) and (B) hold, it suffices to show (C) and (D).

We prove (C) and (D) by induction on i. Obviously, G_0 satisfies (C) and (D). Suppose (C) and (D) hold for $i = j$. Let P_j be a proper ear of G_j from s_j to t_j. Then the reverse path $\bar{P}_j$ exists by the induction hypothesis of the condition (D). Consider $G_{j+1} = G_j + P_j + \bar{P}_j$.

We first show that the condition (C) holds for $i = j + 1$.

1. Suppose $u, v \in V_j$. Then it follows from the induction hypothesis that there exist both an elementary even path and an elementary odd path from u to v in G_{j+1}.

2. Suppose $u \in V_j$ and $v \in V_{j+1} \setminus V_j$. Let P' be a path from s_j to v along P_j. Since there exist both an elementary even path P_e and an elementary odd path P_o from u to s_j in G_j, one of $P_\mathrm{e} \cdot P'$ and $P_\mathrm{o} \cdot P'$ is an elementary even path, and the other is an elementary odd path. If $v \in V_j$ and $u \in V_{j+1} \setminus V_j$, we can prove that there exist both an elementary even path and an elementary odd path from u to v in a similar way.

3. Suppose $u, v \in V_{j+1} \setminus V_j$. Without loss of generality we assume s_j, v, u, and t_j appear on P_j in this order. Let P' be a path from u to t_j along P_j, and P'' a path from s_j to v along P_j. Since there exist both an elementary even path P_{e} and an elementary odd path P_{o} from t_j to s_j in G_j, one of $P' \cdot P_{\mathrm{e}} \cdot P''$ and $P' \cdot P_{\mathrm{o}} \cdot P''$ is an elementary even path, and the other is an elementary odd path.

Thus there exist both an elementary even path and an elementary odd path from u to v in G_{j+1} for every vertex pair $u, v \in V_{j+1}$.

We next show that the condition (D) holds for $i = j + 1$. Let P be a proper ear of G_{j+1} from s to t. Since there exist both an elementary even path P_{e} and an elementary odd path P_{o} from t to s in G_{j+1}, either $P \cdot P_{\mathrm{e}}$ or $P \cdot P_{\mathrm{o}}$ is an elementary odd cycle. Hence every arc of P has the reverse arc. □

We are now ready to prove Theorem 1. The necessity is obvious, as bipartite digraphs have no odd cycles. To prove the sufficiency, assume that a digraph G is odd-cycle-symmetric. If G is not bipartite, then it is symmetric by Proposition 4, which completes the proof of Theorem 1.

Theorem 1 leads to a linear time algorithm for recognizing odd-cycle-symmetry of a digraph as follows.

Corollary 5. *Given a digraph G, we can determine whether G is odd-cycle-symmetric in $\mathrm{O}(m+n)$ time, where m and n are the numbers of the arcs and vertices, respectively.*

Proof. We can decompose G into strongly connected components in linear time [8]. We can also decompose each strongly connected component into the components whose underlying graphs are 2-connected in linear time [6, 8]. Note that these components are also strongly connected, and hence cycle-connected. Since every cycle in G is contained in some component, G is odd-cycle-symmetric if and only if every component is odd-cycle-symmetric. By Theorem 1, it suffices to check if every obtained component D is bipartite or symmetric. We can check whether D is bipartite or not in $\mathrm{O}(m_D + n_D)$ time, where m_D and n_D represent the numbers of the arcs and vertices of D, respectively. Checking the symmetry of D requires $\mathrm{O}(m_D + n_D)$ time. Thus we can determine whether G is odd-cycle-symmetric or not in $\mathrm{O}(m+n)$ time. □

3 Odd-Cycle-Symmetry of Weighted Digraphs

Let $G = (V, A)$ be an odd-cycle-symmetric digraph, and w be a weight function defined on the arc set A. We write $w(P) = \Sigma_{a \in P} w(a)$ for a path P, and $w(C) = \Sigma_{a \in C} w(a)$ for a cycle C. Then (G, w) is said to be *odd-cycle-symmetric* if w satisfies that $w(C) = w(\bar{C})$ for every elementary odd cycle C.

If an arc a has the reverse arc $\bar{a}$, then $d(a)$ denotes $w(a) - w(\bar{a})$. Note that $w(C) = w(\bar{C})$ for a cycle C is equivalent to $d(C) = 0$.

Theorem 6. *Let $G = (V, A)$ be a cycle-connected symmetric digraph that is not bipartite. A weighted digraph (G, w) is odd-cycle-symmetric if and only if there exists a function p on V such that $d(a) = p(v) - p(u)$ for each arc $a = (u, v) \in A$.*

Proof. The necessity is obvious, as the existence of such a function p implies that $w(C') = w(\bar{C}')$ for every elementary cycle C'. To prove the sufficiency, we assume that (G, w) is odd-cycle-symmetric. A digraph G has an elementary odd cycle C by Lemma 3.

Let $G_0, G_1, \ldots, G_k = G$ be a sequence of subgraphs of G which satisfies the following conditions.

1. The subgraph G_0 consists of C and $\bar{C}$.

2. For $i = 0, 1, \ldots, k-1$, $G_{i+1} = G_i + P_i + \bar{P}_i$, where P_i is a proper ear of G_i from s_i to t_i.

Proposition 4 guarantees the existence of such a sequence.

We show by induction on i that $G_i = (V_i, A_i)$ has a function p_i on V_i such that $d(a) = p_i(v) - p_i(u)$ for each arc $a = (u, v) \in A_i$. It is trivial that G_0 satisfies this property. Suppose G_j satisfies the property. Consider $G_{j+1} = G_j + P_j + \bar{P}_j$.

By Proposition 4, there exist both an elementary even path P_{e} and an elementary odd path P_{o} from t_j to s_j in G_j, and either $P_j \cdot P_{\mathrm{e}}$ or $P_j \cdot P_{\mathrm{o}}$ is an elementary odd cycle. Furthermore, P_{e} and P_{o} satisfy $d(P_{\mathrm{e}}) = d(P_{\mathrm{o}}) = p_j(s_j) - p_j(t_j)$. Thus we have $d(P_j) = p_j(t_j) - p_j(s_j)$. We define a function p_{j+1} on V_{j+1} as follows. If $v \in V_j$, then we set $p_{j+1}(v) = p_j(v)$. Otherwise, we set $p_{j+1}(v) = p_j(s_j) + d(P_v)$, where P_v is a path from s_j to v along P_j. Then $d(a) = p_{j+1}(v) - p_{j+1}(u)$ holds for each arc $a = (u, v) \in A_{j+1}$. Thus G_{j+1} satisfies the property.

Since $G = G_k$, there exists a function p on V such that $d(a) = p(v) - p(u)$ for each arc $a = (u, v) \in A$. □

Corollary 7. *Given a weighted digraph (G, w), we can determine whether (G, w) is odd-cycle-symmetric in $\mathrm{O}(m + n)$ time, where m and n are the numbers of the arcs and vertices, respectively.*

Proof. As in the the proof of Corollary 5, recognizing odd-cycle-symmetry of (G, w) can be reduced to that of cycle-connected digraphs in linear time. Hence we may assume $G = (V, A)$ is cycle-connected. A cycle-connected weighted digraph (G, w) is odd-cycle-symmetric if and only if G is bipartite, or G is a symmetric graph with a function p such that $d(a) = p(v) - p(u)$ for each arc $a = (u, v) \in A$. Whether G is bipartite or not can be checked in linear time. It also takes a linear time to check whether G is symmetric or not. The existence of the function p can be checked in $\mathrm{O}(m + n)$ time as follows. Take a vertex $r \in V$ and find a directed spanning tree T in G rooted at r. For each vertex $v \in V$, let P_v be the unique path from r to v in T, and set $p(v) := d(P_v)$. Note that $p(r) = 0$. Then for each arc $a = (u, v) \in A \setminus T$, check if $d(a) = p(v) - p(u)$ holds.

Thus we can determine whether (G, w) is odd-cycle-symmetric or not in linear time. □

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