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Naonori KAKIMURA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

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Abstract

Kim, Kojima, Mevissen, and Yamashita recently proved that a chordal-structured positive semidefinite (PSD) matrix can be decomposed as a sum of PSD matrices that correspond to the maximal cliques. Their proof is based on a characterization for PSD matrix completion of a chordal-structured matrix due to Grone, Johnson, Sá, and Wolkowicz in 1984. This note gives a direct and simpler proof for the result of Kim *et al.*, which leads to an alternative proof of Grone *et al.*

1 Matrix Decomposition of Chordal-Structured Matrices

Let $G = (N, E)$ be a graph with vertex set $N = \{1, \ldots, n\}$ and edge set $E \subseteq N \times N$. In this note, we regard G as an undirected graph by assuming that (i, j) and (j, i) are identified, i.e., if $(i, j) \in E$ then $(j, i) \in E$. A graph G is *chordal* if every cycle of length at least three has a chord. A vertex of G is called *simplicial* if its neighbor forms a clique. A chordal graph is known to have a simplicial vertex. See [1] for basic properties on chordal graphs.

Let $\mathbb{S}^n$ be the set of symmetric matrices of order n, and $\mathbb{S}^n_+$ be the set of positive semidefinite matrices in $\mathbb{S}^n$. For $F \subseteq N \times N$, we define $\mathbb{R}_F = \{X \in \mathbb{S}^n \mid X_{ij} = 0, \forall (i, j) \notin F\}$. For a graph $G = (N, E)$, let $\mathbb{S}^n_+(E, 0)$ be the set of PSD matrices all of whose entries not in E are equal to zero. Thus $\mathbb{S}^n_+(E, 0) = \mathbb{S}^n_+ \cap \mathbb{R}_E$. Note that $\mathbb{S}^n_+(E, 0)$ is a closed set, because $\mathbb{S}^n_+$ and $\mathbb{R}_E$ are closed. For a set $W \subseteq N$, we denote $\mathbb{S}^n_+(W) = \mathbb{S}^n_+ \cap \mathbb{R}_{W\times W}$.

Kim, Kojima, Mevissen, and Yamashita [3] showed that a chordal-structured PSD matrix can be decomposed as a sum of PSD matrices that correspond to the maximal cliques. This note gives an alternative proof using simple linear algebra, which allows us to impose an additional rank condition on PSD matrices obtained by the decomposition. We denote $E_{\mathrm{d}} = \{(i, i) \mid i \in N\}$.

*Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. `kakimura@mist.i.u-tokyo.ac.jp`

Theorem 1. *Let $G = (N, E)$ be a chordal graph with $E_{\mathrm{d}} \subseteq E$, and $C_1, \ldots, C_p$ be the maximal cliques of G. For a symmetric matrix $A \in \mathbb{R}_E$ of order n, the following* (a) *and* (b) *are equivalent.*

(a) *The matrix A is positive semidefinite.*

(b) *There exist $Y^k \in \mathbb{S}^n_+(C_k)\,(k = 1, \ldots, p)$ such that $A = \sum_{k=1}^p Y^k$ and $\mathrm{rank} A = \sum_{k=1}^p \mathrm{rank} Y^k$.*

Proof. It suffices to show "(a)⇒(b)". Assume that A is positive semidefinite.

We prove this statement by induction on n. The case of $n = 1$ is obvious. Assume that $n > 1$. Since G is chordal, G has a simplicial vertex v. We may suppose that the maximal clique containing v, which is unique, is C_1. By row and column permutations, we may further suppose that $v = 1$ and $C_1 \setminus \{v\} = \{2, \ldots, m\}$. Thus the matrix A is in the form of

$$A = \begin{pmatrix} a_{11} & \boldsymbol{a_1}^{\mathrm{T}} & 0 \\ \boldsymbol{a_1} & A[C_1', C_1'] & A[C_1', \overline{C}_1] \\ 0 & A[\overline{C}_1, C_1'] & A[\overline{C}_1, \overline{C}_1] \end{pmatrix},$$

where $C_1' = C_1 \setminus \{v\}$, $\overline{C}_1 = N \setminus C_1$, and $A[I, J]$ in general denotes the submatrix with row set I and column set J.

First assume that $a_{11} = 0$. Since A is positive semidefinite, we have $\boldsymbol{a_1} = 0$. The subgraph G' induced by $N \setminus \{1\}$ is chordal, and the maximal cliques of G' are those in $C_1', C_2, \ldots, C_p$. Therefore, by applying the induction hypothesis to the matrix obtained from A by deleting the first row and column, A can be represented by $A = \sum_{k=1}^p Y^k$ for some $Y^1 \in \mathbb{S}^n_+(C_1')$ and $Y^k \in \mathbb{S}^n_+(C_k)\,(k = 2, \ldots, p)$ with $\sum_{k=1}^p \mathrm{rank} Y^k = \mathrm{rank} A$. By $Y^1 \in \mathbb{S}^n_+(C_1)$, the condition (b) holds.

Next assume that $a_{11} \neq 0$. Then the matrix A can be transformed into

$$LAL^{\mathrm{T}} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & A[C_1', C_1'] - a_{11}^{-1}\boldsymbol{a_1}\boldsymbol{a_1}^{\mathrm{T}} & A[C_1', \overline{C}_1] \\ 0 & A[\overline{C}_1, C_1'] & A[\overline{C}_1, \overline{C}_1] \end{pmatrix},$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -a_{11}^{-1}\boldsymbol{a_1} & I & O \\ 0 & O & I \end{pmatrix}.$$

Hence

$$\begin{aligned} A &= L^{-1}\left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & O & O \\ 0 & O & O \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & A[C_1', C_1'] - a_{11}^{-1}\boldsymbol{a_1}\boldsymbol{a_1}^{\mathrm{T}} & A[C_1', \overline{C}_1] \\ 0 & A[\overline{C}_1, C_1'] & A[\overline{C}_1, \overline{C}_1] \end{pmatrix} \right\} (L^{-1})^{\mathrm{T}} \\ &= \begin{pmatrix} a_{11} & \boldsymbol{a_1}^{\mathrm{T}} & 0 \\ \boldsymbol{a_1} & a_{11}^{-1}\boldsymbol{a_1}\boldsymbol{a_1}^{\mathrm{T}} & O \\ 0 & O & O \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & A[C_1', C_1'] - a_{11}^{-1}\boldsymbol{a_1}\boldsymbol{a_1}^{\mathrm{T}} & A[C_1', \overline{C}_1] \\ 0 & A[\overline{C}_1, C_1'] & A[\overline{C}_1, \overline{C}_1] \end{pmatrix}. \end{aligned}$$

The first and second matrices in the right-hand side are denoted by X and A', respectively. Since L is nonsingular and A is positive semidefinite, X and A' are both positive semidefinite. The vertex v is simplicial, which implies that if an (i,j)-entry of $a_{11}^{-1}\boldsymbol{a_1}\boldsymbol{a_1}^{\mathrm{T}}$ is nonzero then $i,j \in C_1$. Hence $X \in \mathbb{S}^n_+(C_1)$ holds, and the lower-right part of A' corresponds to the chordal subgraph induced by $N\backslash\{1\}$. The maximal cliques of G' are either $C_2,\dots,C_p$ or $C'_1, C_2,\dots,C_p$. By applying the induction hypothesis to the matrix obtained from A' by deleting the first row and column, A' can be decomposed into $A' = \sum_{k=1}^p Y'^k$ for some $Y'^1 \in \mathbb{S}^n_+(C'_1)$ and $Y'^k \in \mathbb{S}^n_+(C_k)\,(k=2,\dots,p)$ with $\mathrm{rank} A' = \sum_{k=1}^p \mathrm{rank} Y'^k$, where $Y'^1 = O$ if C'_1 is not maximal. Define $Y^1 = X + Y'^1 \in \mathbb{S}^n_+(C_1)$ and $Y^k = Y'^k \in \mathbb{S}^n_+(C_k)\,(k=2,\dots,p)$. Then $A = X + A' = \sum_{k=1}^p Y^k$ holds. Moreover, $\mathrm{rank} A = \mathrm{rank} LAL^{\mathrm{T}} = \mathrm{rank} A' + 1$ and $\mathrm{rank} Y^1 = \mathrm{rank} LY^1L^{\mathrm{T}} = 1 + \mathrm{rank} Y'^1$ hold, which implies that $\sum_{k=1}^p \mathrm{rank} Y^k = \mathrm{rank} A$. Thus the condition (b) holds. $\square$

Note that $\mathrm{rank}(X+Y) \le \mathrm{rank} X + \mathrm{rank} Y$ for matrices X and Y. Hence $Y^k\,(k=1,\dots,p)$ in Theorem 1 are matrices minimizing $\sum_{k=1}^p \mathrm{rank} Y^k$ subject to $\sum_{k=1}^p Y^k = A$ and $Y^k \in \mathbb{S}^n_+(C_k)\,(k=1,\dots,p)$.

2 PSD Matrix Completion for Chordal-Structured Matrices

Positive semidefinite matrix completion is the following problem: Given a symmetric matrix A some of whose entries, denoted by $E \subseteq N \times N$, are specified, can we make A positive semidefinite by assigning values to entries not in E?

We define $\mathbb{S}^n_+(E,?) \subseteq \mathbb{S}^n$ to be the set of symmetric matrices that can be made positive semidefinite by changing entries in $\overline{E}$, where $\overline{F}$ is the complement for a set $F \subseteq N \times N$. That is, $X \in \mathbb{S}^n_+(E,?)$ means that there exists $X' \in \mathbb{S}^n_+$ such that $X'_{ij} = X_{ij}$ for any $(i,j) \in E$. Thus $\mathbb{S}^n_+(E,?)$ represents the set of symmetric matrices that can be completed to a positive semidefinite matrix. By the definition, we have $\mathbb{S}^n_+(E,?) = \mathbb{S}^n_+ + \mathbb{R}_{\overline{E}}$ (Minkowski sum), which is a closed set as shown in Lemma 5 below.

For a convex cone $\mathbb{K} \subseteq \mathbb{S}^n$, we define the *polar* $\mathbb{K}^*$ to be $\mathbb{K}^* = \{Y \mid X \bullet Y \ge 0, \forall X \in \mathbb{K}\}$, where $X \bullet Y$ is the trace of XY for $X, Y \in \mathbb{S}^n$. Taking the polar of $\mathbb{S}^n_+(E,0)$ in Theorem 1, we obtain Corollary 2 below, which is equivalent to the result of Grone *et al.* [2] about the positive semidefinite matrix completion for chordal-structured matrices. Kim *et al.* [3] derived Theorem 1 from this corollary. Thus Theorem 1 and Corollary 2 are equivalent in terms of polarity.

Corollary 2 (Grone, Johnson, Sá, and Wolkowicz [2]). *Let $G = (N,E)$ be a chordal graph with $E_{\mathrm{d}} \subseteq E$, and $C_1,\dots,C_p$ be the maximal cliques of G. Then the following (a) and (b) are equivalent.*

(a) *The matrix A is in $\mathbb{S}^n_+(E,?)$.*

(b) *For any $k \in \{1,\dots,p\}$, the principal submatrix with row and column sets C_k is positive semidefinite.*

In order to derive this corollary from Theorem 1, we use the following well-known facts.

Lemma 3. (1) *For two cones* $\mathbb{K}_1, \mathbb{K}_2$*, it holds that* $(\mathbb{K}_1 + \mathbb{K}_2)^* = \mathbb{K}_1^* \cap \mathbb{K}_2^*$.

(2) *For two closed convex cones* $\mathbb{K}_1, \mathbb{K}_2$*, it holds that* $(\mathbb{K}_1 \cap \mathbb{K}_2)^* = \mathrm{cl}(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

(3) *It holds that* $(\mathbb{S}^n_+)^* = \mathbb{S}^n_+$ *and* $(\mathbb{R}_F)^* = \mathbb{R}_{\overline{F}}$ *for any* $F \subseteq N \times N$.

Proof of Corollary 2. It suffices to show "(b)⇒(a)".

By Theorem 1, $\mathbb{S}^n_+(E, 0) = \mathbb{S}^n_+(C_1) + \cdots + \mathbb{S}^n_+(C_p)$. Since $\mathbb{S}^n_+(E, 0) = \mathbb{S}^n_+ \cap \mathbb{R}_E$, we obtain

$$\mathrm{cl}(\mathbb{S}^n_+ + \mathbb{R}_{\overline{E}}) = (\mathbb{S}^n_+(C_1))^* \cap \cdots \cap (\mathbb{S}^n_+(C_p))^*$$

by taking the polarity and Lemma 3. The left-hand side is equal to $\mathbb{S}^n_+ + \mathbb{R}_{\overline{E}} = \mathbb{S}^n_+(E, ?)$ since $\mathbb{S}^n_+(E, ?)$ is closed. Since $\mathbb{S}^n_+(C_k) = \mathbb{S}^n_+ \cap \mathbb{R}_{D_k}$, where $D_k = C_k \times C_k$, for any k, Lemma 3 implies that $(\mathbb{S}^n_+(C_k))^* = \mathrm{cl}(\mathbb{S}^n_+ + \mathbb{R}_{\overline{D}_k})$. Therefore, $\mathbb{S}^n_+(E, ?) \supseteq \bigcap_{k=1}^p (\mathbb{S}^n_+ + \mathbb{R}_{\overline{D}_k})$ holds. This right-hand side means that the principal submatrix with row and column sets C_k is positive semidefinite for any k, because $\mathbb{S}^n_+ + \mathbb{R}_{\overline{D}_k} = \mathbb{S}^{|C_k|}_+ \oplus \mathbb{R}_{\overline{D}_k}$. Thus we have "(b)⇒(a)". □

It remains to show that $\mathbb{S}^n_+(E, ?)$ is closed. For that purpose, we need the following lemma.

Lemma 4 (Corollary 9.1.3 of Rockafellar [4])**.** *Let* $\mathbb{K}_1, \mathbb{K}_2$ *be two closed convex cones, and assume that, if* $X_1 \in \mathbb{K}_1$ *and* $X_2 \in \mathbb{K}_2$ *satisfy* $X_1 + X_2 = 0$*, then we have* $X_i \in \mathbb{K}_i \cap (-\mathbb{K}_i)$ $(i = 1, 2)$ *holds. Then* $\mathrm{cl}(\mathbb{K}_1 + \mathbb{K}_2) = \mathbb{K}_1 + \mathbb{K}_2$ *holds.*

Lemma 5. *Let* $G = (N, E)$ *be a graph with* $E_\mathrm{d} \subseteq E$*. Then* $\mathbb{S}^n_+(E, ?)$ *is a closed set.*

Proof. Note that $\mathbb{S}^n_+(E, ?) = \mathbb{S}^n_+ + \mathbb{R}_{\overline{E}}$. Assume that $A_1 \in \mathbb{S}^n_+$ and $A_2 \in \mathbb{R}_{\overline{E}}$ satisfy $A_1 + A_2 = O$. Then $A_2 \in \mathbb{R}_{\overline{E}}$ implies $(A_1)_{ij} = (A_2)_{ij} = 0$ for any $(i, j) \in E$. By $E_\mathrm{d} \subseteq E$ and $A_1 \in \mathbb{S}^n_+$, we have $A_1 = O$. Hence $A_1 = A_2 = O$ holds. Since O is contained in $\mathbb{S}^n_+ \cap (-\mathbb{S}^n_+)$ and $\mathbb{R}_{\overline{E}} \cap (-\mathbb{R}_{\overline{E}})$, it follows from Lemma 4 that $\mathrm{cl}(\mathbb{S}^n_+ + \mathbb{R}_{\overline{E}}) = \mathbb{S}^n_+ + \mathbb{R}_{\overline{E}}$. Thus $\mathbb{S}^n_+(E, ?)$ is closed. □

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